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Supersymmetry and point canonical transformations in the path integral

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Abstract. We show that in the path integral quantisation procedure for free non-relativistic particles, point canonical transformations can be performed by a naive change of variables in the path integral, provided we use the corresponding supersymmetric path integral.

1. Introduction

For the path integral treatment of simple quantum mechanics it is known that point canonical transformations performed by simply changing integration variables in the path integral leads to erroneous results. Edwards and Gulyaev [1] considered the simple problem of the quantum mechanics of a free non-relativistic particle of unit mass moving in two dimensions. They showed that on transforming from cartesian to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the path integral

$$\int \mathrm{D}x\,\mathrm{D}y \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t[\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)]\right) \tag{1.1}$$

is not equal to

$$\int \mathrm{D}r\,\mathrm{D}\theta J[r] \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t[\tfrac{1}{2}(\dot{r}^2+r^2\dot{\theta}^2)]\right) \tag{1.2}$$

where $J[r] \sim \prod_i r_i$. The reason the above path integral gives the wrong result can be traced to the stochastic nature of the quantum paths. For ordinary differential paths upon discretisation $[(\Delta x)^2 + (\Delta y)^2]/\Delta t$ is negligible as $\Delta t \to 0$, whereas for the quantum stochastic paths $[(\Delta x)^2+(\Delta y)^2]/\Delta t = \mathrm{O}(\hbar)$. In order to make equation (1.2) correct an additional potential term of order $\mathrm{O}(\hbar^2 r^{-2})$ is required. See e.g. [2–4].

Gervais and Jevicki [5] discussed the problem of point canonical transformations in the path integral method with the conclusion that for the case of a trivial system such as $L = \frac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b$ a careful treatment of point transformations by the use of the discretised path integral leads to additional potential terms in the transformed action, contrary to the case of the usual formal treatment. Omote [6] was able to generalise the result of Gervais and Jevicki to systems with $L = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j$ on curved manifolds.

For the system with Lagrangian $L = \frac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b$ considered by Gervais and Jevicki, we write the transformed path integral in curvilinear coordinates q^μ as

$$\int D[x]J_x \exp\left(\frac{i}{\hbar}\int dt\{\tfrac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b\}\right) = \int D[q]J_q \exp\left(\frac{i}{\hbar}\int dt\{\tfrac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu\}\right) \tag{1.3}$$

where $J_x = 1$ and J_q are multiplicative Jacobian factors. Naive formal path integral methods would give the erroneous result that $J_q = 1$, but in this case it is equal to the exponential of the additional potential terms.

We can make an analogy with statistical mechanics [7]. In statistical mechanics the ensemble average is taken over phase space $\int\prod dq^n dP_n \ldots$. However, Jacobian factors will appear if instead of phase space the average is taken over coordinate space; $\int\prod dq^n dP_n \ldots$ becomes $\int\prod J dq^n \ldots$. The appearance of a Jacobian factor in the transformed path integral implies that we should consider the integration to be performed in a bigger space in which the multiplicative factor is a constant.

In this paper we propose a new scheme for dealing with the problem of point canonical transformations of the path integral. We write the Jacobian factors of equation (1.3) as fermionic path integrals. When this is done, naive change of bosonic and fermionic variables in the path integral no longer leads to erroneous results. The resulting path integral will also be invariant under additional transformations which mix bosonic and fermionic variables; the path integral becomes supersymmetric.

In [8] it was shown that the formal path integral containing bosonic and fermionic variables is the continuum limit of the discrete mid-point path integral. Thus in the following sections the discrete path integrals are evaluated using the mid-point prescription.

The paper is organised as follows. In section 2 we briefly review the discretised path integral used by Gervais and Jevicki [5]. We then take the continuum limit and give an explicit expression for the extra potential terms. The Jacobian factor will be expressed as a fermionic functional integral. In section 3 we use the result of Omote [6] to generalise our discussion of section 2 to include the case of curved manifolds. Finally in section 4 we give a summary of our results. In the appendix we generalise our discussion to include non-relativistic particles moving in a potential.

2. Free particle on a flat manifold

The Lagrangian for a free particle moving on an N-dimensional Euclidean manifold is given by $L = \frac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b$. We consider a general point canonical transformation from cartesian coordinates x^a to curvilinear coordinates q^α given by

$$x^a(t) = F^a[q(t)]. \tag{2.1}$$

The short time kernel connecting states at $x^a(t_i) = x^a_i$ to states at $x^a(t_{i+1}) = x^a_{i+1}$, where $t_{i+1} = t_i + \Delta t$, is given by the discrete path integral

$$K[x_{i+1}, t_{i+1}; x_i, t_i] = \int \prod_{a=1}^{N} \left\{\frac{dx^a_i}{(2\pi i\hbar\Delta t)^{1/2}}\right\} \exp\left(\frac{i}{\hbar}\Delta t \tfrac{1}{2}\delta_{ab}\frac{\Delta x^a_i}{\Delta t}\frac{\Delta x^b_i}{\Delta t}\right) \tag{2.2}$$

where $\Delta x^a_i = x^a_{i+1} - x^a_i$. Performing the point canonical transformation and expanding about the mid-point $\bar{q}^\mu_i = \frac{1}{2}q^\mu_{i+1} + \frac{1}{2}q^\mu_i$, we have

$$\Delta x^a_i = e^a_\mu(\bar{q}_i)\Delta q^\mu_i + \tfrac{1}{24}\{\partial_\rho\partial_\sigma e^a_\mu(\bar{q}_i)\}\Delta q^\rho_i \Delta q^\sigma_i \Delta q^\mu_i + \ldots \tag{2.3}$$

where $e^a_\mu[q(t)] = \partial_\mu F^a[q(t)]$. Naively we would expect $\Delta x^a_i = e^a_\mu(\bar{q}_i)\Delta q^\mu_i$ and then the transformed action would be

$$A[q_{i+1}, q_i] = \Delta t\left[\tfrac{1}{2}g_{\mu\nu}(\bar{q}_i)\frac{\Delta q^\mu_i}{\Delta t}\frac{\Delta q^\nu_i}{\Delta t}\right] \tag{2.4}$$

where $g_{\mu\nu}(\bar{q}_i) = \delta_{ab}e^a_\mu(\bar{q}_i)e^b_\nu(\bar{q}_i)$. This is equivalent to a naive change of variables and is incorrect. Due to the stochastic nature of the quantum paths we find that $(\Delta q)^3/\Delta t$ and $(\Delta q)^4/\Delta t$ still contribute to the path integral and thus cannot be neglected. The full contribution to the action is

$$A[q_{i+1}, q_i] = \Delta t\left[\tfrac{1}{2}g_{\mu\nu}(\bar{q}_i)\frac{\Delta q^\mu_i}{\Delta t}\frac{\Delta q^\nu_i}{\Delta t} + \tfrac{1}{24}\delta_{ab}e^b_\nu(\bar{q}_i)\{\partial_\rho\partial_\tau e^a_\mu(\bar{q}_i)\}\frac{\Delta q^\mu_i}{\Delta t}\frac{\Delta q^\nu_i}{\Delta t}\Delta q^\rho_i\Delta q^\tau_i\right]. \tag{2.5}$$

The transformed integration measure is

$$\prod_{a=1}^{N}\mathrm{d}x^a_i = g^{1/2}(q_i)\prod_{\mu=1}^{N}\mathrm{d}q^\mu_i \tag{2.6}$$

$$= g^{1/2}(\bar{q}_i)\exp(-\tfrac{1}{2}\Gamma^\alpha_{\mu\alpha}(\bar{q}_i)\Delta q^\mu_i)\prod_{\mu=1}^{N}\mathrm{d}q^\mu_i \tag{2.7}$$

where $g(q_i) = \det g_{\mu\nu}(q_i)$, $\Gamma^\alpha_{\mu\beta} = \frac{1}{2}g^{\alpha\lambda}[\partial_\mu g_{\lambda\beta} + \partial_\beta g_{\mu\lambda} - \partial_\lambda g_{\mu\beta}]$, and we have expanded $g^{1/2}(q_i)$ about the mid-point. Note that the exponential term in equation (2.7) was not found by Gervais and Jevicki [5]. This is because although they defined the transformed integration measure by equation (2.6), they later used a non-equivalent definition.

Using the arguments outlined in [5] for replacing the second term on the right of equation (2.5) by a potential term we obtain the short time kernel

$$\int\prod_{\mu=1}^{N}\left\{\frac{\mathrm{d}q^\mu_i}{(2\pi\mathrm{i}\hbar\Delta t)^{1/2}}\right\}g^{1/2}(\bar{q}_i)\exp\left(\frac{\mathrm{i}}{\hbar}\Delta t\left\{\tfrac{1}{2}g_{\mu\nu}(\bar{q}_i)\frac{\Delta q^\mu_i}{\Delta t}\frac{\Delta q^\nu_i}{\Delta t} + \mathrm{i}\hbar V_{\mathrm{v}} - \hbar^2 V_e\right\}\right) \tag{2.8}$$

where

$$V_{\mathrm{v}} = \tfrac{1}{2}\Gamma^\alpha_{\mu\alpha}(\bar{q}_i)\frac{\Delta q^\mu_i}{\Delta t} \tag{2.9}$$

$$V_e = \tfrac{1}{8}g^{\mu\nu}(\bar{q}_i)\Gamma^\beta_{\mu\alpha}(\bar{q}_i)\Gamma^\alpha_{\nu\beta}(\bar{q}_i). \tag{2.10}$$

Finally, taking the continuum limit we obtain the transformed path integral in curvilinear coordinates:

$$\int\mathrm{D}[q]\exp\left(\frac{\mathrm{i}}{\hbar}\int\mathrm{d}t\{\tfrac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu + \mathrm{i}\hbar V_{\mathrm{v}} - \hbar^2 V_e\}\right). \tag{2.11}$$

We would like to interpret the extra potential terms in equation (2.11) as a Jacobian factor

$$J_q = \exp\left(\frac{\mathrm{i}}{\hbar}\int\mathrm{d}t\{\mathrm{i}\hbar V_{\mathrm{v}} - \hbar^2 V_e\}\right) \tag{2.12}$$

and from the transformation of the path integral given by equation (1.3), we would like J_q to be related to J_x by a point canonical transformation.

For finite-dimensional manifolds

$$\int D[\xi^*]\,D[\xi]\exp\left(\frac{i}{\hbar}\int dt\{i\xi^{*a}\dot{\xi}_a\}\right) = 1 \tag{2.13}$$

thus this is a reasonable candidate for J_x. We now need to give the transformation of the fermionic variables which is compatible with the point canonical transformation of the bosonic variables.

For two overlapping coordinate patches on a manifold M with $x^a \in \bar{U}$ and $q^\mu \in U$, we define $e^a_\alpha = \partial x^a/\partial q^\alpha$ and $E^\alpha_a = \partial q^\alpha/\partial x^a$. Then

$$\bar{g}_{ab}e^a_\mu e^b_\nu = g_{\mu\nu} \tag{2.14}$$

$$\partial_\mu E^\nu_a = -\Gamma^\nu_{\mu\lambda}E^\lambda_a + \bar{\Gamma}^b_{ca}e^c_\mu E^\nu_b \tag{2.15}$$

$$\partial_\mu e^a_\nu = \Gamma^\lambda_{\mu\nu}e^a_\lambda - \bar{\Gamma}^a_{cb}e^c_\mu e^b_\nu \tag{2.16}$$

where $\Gamma^\mu_{\nu\beta}$ is a connection in U and $\bar{\Gamma}^b_{cd}$ is a connection in $\bar{U}$. A general set of point canonical transformations for both bosonic and fermionic variables can be defined by

$$\begin{aligned}
\dot{x}^a &= e^a_\mu \dot{q}^\mu \\
\xi^{*a} &= e^a_\mu \psi^{*\mu} \\
\xi_b &= E^\nu_b \psi_\nu \\
\bar{R}_a{}^b{}_c{}^d &= E^\alpha_a e^b_\beta E^\gamma_c e^d_\delta R_\alpha{}^\beta{}_\gamma{}^\delta
\end{aligned} \tag{2.17}$$

where $\psi^{*\mu}$ and ψ_μ are the transformed fermionic variables and $\bar{R}_a{}^b{}_c{}^d$ is the Riemann curvature tensor.

Returning to our problem of transforming from cartesian coordinates to curvilinear coordinates, in which case $\bar{g}_{ab} = \delta_{ab}$, the connections $\bar{\Gamma}^b_{ca}$ and the curvature tensors $\bar{R}_a{}^b{}_c{}^d$ are zero. Hence the transformed Jacobian factor is

$$\begin{aligned}
J_x &= \int D[\xi^*]\,D[\xi]\exp\left(\frac{i}{\hbar}\int dt\{i\xi^{*a}\dot{\xi}_a\}\right) \to \\
J_q &= \int D[\psi^*]\,D[\psi]\exp\left(\frac{i}{\hbar}\int dt\left\{\psi^{*\mu}\left[i\frac{d}{dt}\delta^\nu_\mu - i\Gamma^\nu_{\mu\alpha}\dot{q}^\alpha\right]\psi_\nu\right\}\right).
\end{aligned} \tag{2.18}$$

It remains to show that integrating out the fermionic variables we obtain the extra potential terms of equation (2.12). To evaluate the fermionic integral it is easiest to work with imaginary time τ. Then

$$J_q = \det\left[\left\{\frac{d}{d\tau}\delta^\nu_\mu - \Gamma^\nu_{\mu\alpha}\dot{q}^\alpha\right\}\delta(\tau-\tau')\right]. \tag{2.19}$$

Using the retarded boundary condition

$$\frac{d}{d\tau}\theta(\tau-\tau') = \delta(\tau-\tau') \tag{2.20}$$

the determinant can be expressed as

$$\begin{aligned} J_q &= \det[\mathrm{d}/\mathrm{d}\tau]\,\det[\delta^\nu_\mu\delta(\tau-\tau') - \theta(\tau-\tau')\Gamma^\nu_{\mu\alpha}\dot{q}^\alpha] \\ &= \exp\mathrm{Tr}\{\ln[\delta^\nu_\mu\delta(\tau-\tau') - \theta(\tau-\tau')\Gamma^\nu_{\mu\alpha}\dot{q}^\alpha]\} \end{aligned} \tag{2.21}$$

where $\det[\mathrm{d}/\mathrm{d}\tau]$ is a constant which can be set equal to one (this shows that $J_x = 1$). The identity $\int \mathrm{d}\tau\,\mathrm{d}\tau'\theta(\tau-\tau')\theta(\tau'-\tau) = 0$ can be used to show that terms higher than second order will not contribute to the expansion of the logarithm in equation (2.21). However, because of the stochastic nature of the quantum paths $\langle\dot{q}^\mu(\tau)\dot{q}^\nu(\tau')\rangle = \hbar g^{\mu\nu}[q(\tau)]\delta(\tau-\tau')$, a second order term will the contribute. Expanding the logarithm the non-zero terms give

$$\begin{aligned} J_q = \exp\bigg(&-\int \mathrm{d}\tau\ \theta(0)\Gamma^\alpha_{\mu\alpha}\dot{q}^\mu \\ &-\frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}\tau'\ \theta(\tau-\tau')\Gamma^\beta_{\mu\alpha}[q(\tau)]\dot{q}^\mu(\tau)\theta(\tau'-\tau)\Gamma^\alpha_{\nu\beta}[q(\tau')]\dot{q}^\nu(\tau') + \ldots\bigg). \end{aligned} \tag{2.22}$$

Using $\theta(0) = \frac{1}{2}$, the stochastic nature of the quantum paths, and changing to real time we obtain the desired result

$$J_q = \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{\mathrm{i}\hbar V_\mathrm{v} - \hbar^2 V_e\}\right). \tag{2.12}$$

In summary the path integral

$$\int \mathrm{D}[x]\,\mathrm{D}[\xi^*]\,\mathrm{D}[\xi]\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{\tfrac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b + \mathrm{i}\xi^{*a}\dot{\xi}_a\}\right) \tag{2.23}$$

remains invariant under the naive change of variables corresponding to a point canonical transformation. The transformed path integral

$$\int \mathrm{D}[q]\,\mathrm{D}[\psi^*]\,\mathrm{D}[\psi]\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{\tfrac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu + \psi^{*\mu}[\mathrm{i}(\mathrm{d}/\mathrm{d}t)\delta^\nu_\mu - \mathrm{i}\Gamma^\nu_{\mu\alpha}\dot{q}^\alpha]\psi_\nu\}\right) \tag{2.24}$$

upon integrating out the fermionic variables gives equation (2.11), the path integral in curvilinear coordinates. Note that the action in equation (2.23) is trivially supersymmetric for the transformations

$$\begin{aligned} \delta_\epsilon x^a &= \epsilon^*\xi^{*a} - \epsilon\delta^{ab}\xi_b \\ \delta_\epsilon\xi^{*a} &= -\mathrm{i}\epsilon\dot{x}^a \\ \delta_\epsilon\xi_b &= \mathrm{i}\epsilon^*\delta_{bc}\dot{x}^c \end{aligned} \tag{2.25}$$

where ϵ^* and ϵ are anticommuting constants. The action in equation (2.24) is supersymmetric with respect to the following transformations:

$$\begin{aligned} \delta_\epsilon q^\mu &= \epsilon^*\psi^{*\mu} - \epsilon g^{\mu\nu}\psi_\nu \\ \delta_\epsilon\psi^{*\mu} &= -\mathrm{i}\epsilon[\dot{q}^\mu + \mathrm{i}g^{\alpha\beta}\Gamma^\mu_{\beta\gamma}\psi_\alpha\psi^{*\gamma}] \\ \delta_\epsilon\psi_\mu &= \mathrm{i}\epsilon^*[g_{\mu\sigma}\dot{q}^\sigma - \mathrm{i}\Gamma^\beta_{\mu\alpha}\psi^{*\alpha}\psi_\beta] - \epsilon g^{\alpha\beta}\Gamma^\gamma_{\mu\beta}\psi_\beta\psi_\gamma. \end{aligned} \tag{2.26}$$

Equations (2.26) are the point canonical transformations of equations (2.25), where $\delta_\epsilon q^\mu = E^\mu_a\delta_\epsilon x^a$.

3. Free particle on a curved manifold

By requiring the corresponding Hamiltonian for a free particle on a curved manifold to be invariant under general point canonical transformations, Omote [6] was able to write the short time kernel $K[q_{i+1}, t_{i+1}; q_i, t_i]$ as a discrete path integral

$$\int \mathrm{d}^N q_i \frac{g^{-1/4}(q_{i+1}) g^{1/4}(q_i) g^{1/2}(\bar{q})}{(2\pi \mathrm{i}\hbar\Delta t)^{N/2}} \exp\left(\frac{\mathrm{i}}{\hbar}\Delta t \left\{ \tfrac{1}{2} g_{\mu\nu}(\bar{q}) \frac{\Delta q^\mu}{\Delta t}\frac{\Delta q^\nu}{\Delta t} - \hbar^2 V_{\mathrm{E}} \right\}\right) \tag{3.1}$$

where the extra potential is

$$V_{\mathrm{E}} = \tfrac{1}{8}[g^{\mu\nu}(\bar{q})\Gamma^\beta_{\mu\alpha}(\bar{q})\Gamma^\alpha_{\nu\beta}(\bar{q}) - R]. \tag{3.2}$$

In equation (3.2) R is the curvature scalar that can be written as

$$R = g^{\mu\nu}\{\partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial_\mu \Gamma^\sigma_{\nu\sigma} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\sigma\mu}\Gamma^\sigma_{\lambda\nu}\}. \tag{3.3}$$

Expanding about the mid-point we have

$$g^{-1/4}(q_{i+1}) g^{1/4}(q_i) = \exp\left(\frac{\mathrm{i}}{\hbar}\Delta t \left\{ \frac{\mathrm{i}\hbar}{2} \Gamma^\alpha_{\mu\alpha}(\bar{q}) \frac{\Delta q^\mu}{\Delta t} \right\}\right) \tag{3.4}$$

thus taking the continuum limit of equation (3.1), we obtain the path integral

$$\int \mathrm{D}[q] \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t \{\tfrac{1}{2} g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu + \mathrm{i}\hbar V_{\mathrm{v}} - \hbar^2 V_{\mathrm{E}}\}\right) \tag{3.5}$$

where

$$V_{\mathrm{v}} = \tfrac{1}{2}\Gamma^\alpha_{\mu\alpha}\dot{q}^\mu. \tag{3.6}$$

Following the procedure of section 2, we write the path integral equation (3.5) as

$$\int \mathrm{D}[q] J_q \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t \{\{1\ 2 g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu\}\right) \tag{3.7}$$

where the extra potential terms are contained in the Jacobian factor. It will be shown that the Jacobian factor can be written as a fermionic path integral and that the total action is supersymmetric. The supersymmetric generalisation of $L = \frac{1}{2} g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu$ is that of the supersymmetric non-linear σ-model [9]

$$L_{\mathrm{S}} = \tfrac{1}{2} g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu + \psi^{*\mu}\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\delta^\nu_\mu - \mathrm{i}\Gamma^\nu_{\mu\alpha}\dot{q}^\alpha\right]\psi_\nu + \tfrac{1}{2} R_\alpha{}^\beta{}_\gamma{}^\delta \psi^{*\alpha}\psi_\beta \psi^{*\gamma}\psi_\delta \tag{3.8}$$

which is supersymmetric under the supersymmetry transformations of equation (2.26). Thus the Jacobian factor is

$$J_q = \int \mathrm{D}[\psi^*]\,\mathrm{D}[\psi] \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t \left\{ \psi^{*\mu}\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\delta^\nu_\mu - \mathrm{i}\Gamma^\nu_{\mu\alpha}\dot{q}^\alpha\right]\psi_\nu + \tfrac{1}{2} R_\alpha{}^\beta{}_\gamma{}^\delta \psi^{*\alpha}\psi_\beta \psi^{*\gamma}\psi_\delta \right\}\right). \tag{3.9}$$

We show that the path integral equation (3.7) is invariant under the point canonical transformations

$$\begin{aligned}
\dot{q}^{\mu} &= E^{\mu}_{a}\dot{x}^{a} \\
\psi^{*\mu} &= E^{\mu}_{a}\xi^{*a} \\
\psi_{\mu} &= e^{b}_{\mu}\xi_{b} \\
R_{\alpha}{}^{\beta}{}_{\gamma}{}^{\delta} &= e^{a}_{\alpha}E^{\beta}_{b}e^{c}_{\gamma}E^{\delta}_{d}\bar{R}_{a}{}^{b}{}_{c}{}^{d}.
\end{aligned} \tag{3.10}$$

The transformed Jacobian factor is

$$J_x = \int \mathrm{D}[\xi^*]\,\mathrm{D}[\xi]\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left\{\xi^{*a}\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\delta^{b}_{a} - \mathrm{i}\bar{\Gamma}^{b}_{ac}\dot{x}^{c}\right]\xi_b + \tfrac{1}{2}\bar{R}_{a}{}^{b}{}_{c}{}^{d}\xi^{*a}\xi_b\xi^{*c}\xi_d\right\}\right) \tag{3.11}$$

and the transformed path integral is

$$\int \mathrm{D}[x]J_x\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{\tfrac{1}{2}\bar{g}_{ab}\dot{x}^{a}\dot{x}^{b}\}\right). \tag{3.12}$$

It only remains for us to show that

$$\begin{aligned}
J_q &= \int \mathrm{D}[\psi^*]\,\mathrm{D}[\psi]\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left\{\psi^{*\mu}\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\delta^{\nu}_{\mu} - \mathrm{i}\Gamma^{\nu}_{\mu\alpha}\dot{q}^{\alpha}\right]\psi_\nu + \tfrac{1}{2}R_{\alpha}{}^{\beta}{}_{\gamma}{}^{\delta}\psi^{*\alpha}\psi_\beta\psi^{*\gamma}\psi_\delta\right\}\right) \\
&= \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{\mathrm{i}\hbar V_{\mathrm{v}} - \hbar^2 V_{\mathrm{E}}\}\right).
\end{aligned} \tag{3.13}$$

The above identity can be shown using standard coordinate space path integral techniques. The generating functional for the free fermionic field is

$$\begin{aligned}
Z_0[\eta^*,\eta] &= \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{\mathrm{i}\psi^{*\mu}\dot{\psi}_{\mu} + \eta^{*\mu}\psi_\mu + \psi^{*\mu}\eta_\mu\}\right) \\
&= \exp\left(-\frac{1}{\hbar}\int \mathrm{d}t\mathrm{d}t'\{\eta^{*\mu}(t)\theta(t-t')\eta_\mu(t')\}\right)
\end{aligned} \tag{3.14}$$

for retarded boundary conditions, where $\eta^{*\mu}$ and η_μ are fermionic sources. Then for $H_{\mathrm{I}} = \mathrm{i}\psi^{*\mu}\Gamma^{\nu}_{\mu\alpha}\dot{q}^{\alpha}\psi_\nu + \frac{1}{2}R_{\alpha}{}^{\beta}{}_{\gamma}{}^{\delta}\psi^{*\alpha}\psi_\beta\psi^{*\gamma}\psi_\delta$, the Jacobian factor can be written as

$$J_q = \exp\left(-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\ H_{\mathrm{I}}[(-\mathrm{i}\hbar\delta/\delta\eta^*),(\mathrm{i}\hbar\delta/\delta\eta)]\right)Z_0[\eta^*,\eta]\Big|_{\eta^*=\eta=0} \tag{3.15}$$

The fermionic propagator is

$$\langle 0|T\psi^{*\alpha}(t)\psi_\beta(t')|0\rangle = -\hbar\delta^{\alpha}_{\beta}\theta(t'-t) \tag{3.16}$$

and from the stochastic nature of the quantum paths we have

$$\langle 0|T\dot{q}^{\mu}(t)\dot{q}^{\nu}(t')|0\rangle = \mathrm{i}\hbar g^{\mu\nu}[q(t)]\delta(t-t'). \tag{3.17}$$

An example of a four-fermion correlation is

$$\langle 0|T\psi^{*\alpha}(t)\psi_\beta(t)\psi^{*\gamma}(t')\psi_\delta(t')|0\rangle = \hbar^2\delta^{\alpha}_{\beta}\delta^{\gamma}_{\delta}\theta(t-t)\theta(t'-t') - \hbar^2\delta^{\alpha}_{\delta}\delta^{\gamma}_{\beta}\theta(t'-t)\theta(t-t'). \tag{3.18}$$

Using the above propagators and the fact that $\int \mathrm{d}t\,\mathrm{d}t'\theta(t-t')\theta(t'-t) = 0$, the only surviving terms in the perturbation expansion give the desired result

$$J_q = \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{\mathrm{i}\hbar V_{\mathrm{v}} - \hbar^2 V_{\mathrm{E}}\}\right). \tag{3.19}$$

4. Summary and discussion

In section 2 we considered a free non-relativistic particle with Lagrangian $L = \frac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b$ moving on a flat manifold. We found that path integral quantisation leads to a path integral which is not invariant under change of variables corresponding to a point canonical transformation. In section 3, for a free particle moving on a curved manifold with Lagrangian $L = \frac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu$, we found that the usual path integral quantisation

$$\int \mathrm{D}[q] \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{\tfrac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu\}\right) \tag{4.1}$$

is not invariant under a point canonical transformation.

It would be rather disappointing to conclude (as in [4]) that point canonical transformations cannot be handled by the formal path integral. In section 2 we found that formal path integral techniques can be used to show that the path integral remains invariant under naive change of variables, provided we are willing to introduce fictitious fermionic variables.

We would now like to speculate on the significance of adding the fermionic variables. For $L_\mathrm{B} = \frac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b$ in cartesian coordinates, quantisation by the path integral method gives

$$\int \mathrm{D}[x] \exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{L_\mathrm{B}\}\right]. \tag{4.2}$$

We suggest the following quantisation procedure for free particles. First add to the free particle Lagrangian L_B a fermionic Lagrangian L_F which makes the total Lagrangian $L_S = L_\mathrm{B} + L_\mathrm{F}$ supersymmetric. Quantisation is then obtained by integrating over all bosonic and fermionic paths

$$\int \mathrm{D}[x]\,\mathrm{D}[\xi^*]\,\mathrm{D}[\xi] \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{L_\mathrm{B} + L_\mathrm{F}\}\right). \tag{4.3}$$

In cartesian coordinates these two quantisation procedures coincide. Nonetheless, only the second procedure of equation (4.3) will transform correctly for a naive changes of variables in the path integral corresponding to a point canonical transformation. For a free particle on a curved manifold with Lagrangian $L_\mathrm{B} = \frac{1}{2}\bar{g}_{ab}\dot{x}^a\dot{x}^b$ the quantisation procedure given by equation (4.3), where $L_\mathrm{F} = \xi^{*a}[\mathrm{i}(\mathrm{d}/\mathrm{d}t)\delta^b_a - \mathrm{i}\bar{\Gamma}^b_{ac}\dot{x}^c]\xi_b + \frac{1}{2}\bar{R}_a{}^b{}_c{}^d\xi^{*a}\xi_b\xi^{*c}\xi_d$, leads to a quantum theory that is invariant under point canonical transformations, whereas the quantisation given by equation (4.2) leads to a non-equivalent quantum theory that is not invariant under point canonical transformations.

The geometrical interpretation of supersymmetric quantum mechanics is that it corresponds to the Laplacian acting on differential forms [9]; this suggests the following geometrical interpretation of the quantisation procedure of equation (4.3). Quantum mechanics is equivalent to a Hamiltonian (Laplacian) acting on functions (0-forms). We enlarge the space of states by generalising the Hamiltonian (Laplacian) so that it acts on p-forms. This step corresponds to using the supersymmetric path integral instead of the ordinary path integral. By construction the Laplacian acting on differential forms is covariant, hence we expect the supersymmetric path integral to be covariant. The original quantum mechanics is then recovered by restricting the states of interest 'physical states' to be 0-forms.

After this work was completed, we received a preprint [11] from de Alfaro and Gavazzi in which a different approach was used to investigate the covariance of the path integral for supersymmetric quantum mechanics.

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Appendix.

In this appendix we outline how the results for the non-relativistic free particle may be generalised to include a potential. The Lagrangian in cartesian coordinates is

$$L = \tfrac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b - V(x). \tag{A.1}$$

It follows that the path integral

$$\int \mathrm{D}[x]\mathrm{D}[\xi^*]\mathrm{D}[\xi]\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{\tfrac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b + \mathrm{i}\xi^{*a}\dot{\xi}_a - V(x)\}\right) \tag{A.2}$$

will transform correctly for a naive change of variables; however, it is no longer supersymmetric. The above path integral can be made supersymmetric, at least formally, as follows. Assume that the Hamiltonian has a ground-state wavefunction ψ_0 with ground-state energy E_0. Then [10]

$$H\psi_0 = -\tfrac{1}{2}\hbar^2\delta^{ab}\partial_a\partial_b\psi_0 + V(x)\psi_0 = E_0\psi_0 \tag{A.3}$$

and

$$V(x) = \tfrac{1}{2}\delta^{ab}(\partial_a U)(\partial_b U) - \tfrac{1}{2}\hbar\delta^{ab}(\partial_a\partial_b U) + E_0 \tag{A.4}$$

where $U = -\hbar\ln\psi_0$. The path integral equation (A.2) then becomes

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\ E_0\right)\int \mathrm{D}[x]\,\mathrm{D}[\xi^*]\,\mathrm{D}[\xi]\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{\tfrac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b - \tfrac{1}{2}\delta^{ab}(\partial_a U)(\partial_b U) + \xi^{*a}[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\delta_a^b - \delta^{cb}(\partial_c\partial_a U)]\xi_b\}\right). \tag{A.5}$$

The action in equation (A.5) is supersymmetric under the following supersymmetry transformations:

$$\begin{aligned}
\delta_\epsilon x^a &= \epsilon^*\xi^{*a} - \epsilon\delta^{ab}\xi_b \\
\delta_\epsilon\xi^{*a} &= -\mathrm{i}\epsilon[\dot{x}^a + \mathrm{i}\delta^{ab}(\partial_b U)] \\
\delta_\epsilon\xi_a &= \mathrm{i}\epsilon^*[\delta_{ac}\dot{x}^c - \mathrm{i}(\partial_a U)].
\end{aligned} \tag{A.6}$$

The transformed path integral in curvilinear coordinates is

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\ E_0\right)\int \mathrm{D}[q]\,\mathrm{D}[\psi^*]\,\mathrm{D}[\psi]\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left\{\tfrac{1}{2}g_{\mu\nu}\dot q^\mu\dot q^\nu - \tfrac{1}{2}g^{\mu\nu}(\nabla_\mu U)(\nabla_\nu U)\right.\right.$$
$$\left.\left.\times\ \psi^{*\mu}\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\delta^\nu_\mu - \mathrm{i}\Gamma^\nu_{\mu\alpha}\dot q^\alpha - g^{\nu\tau}(\nabla_\mu\nabla_\tau U)\right]\psi_\nu\right\}\right) \tag{A.7}$$

where we have used $(\partial_a U) = E^\mu_a(\partial_\mu U)$, $\nabla_\mu U = \partial_\mu U$, and $\nabla_\mu\nabla_\nu U = (\partial_\mu\partial_\nu U) - \Gamma^\lambda_{\mu\nu}(\partial_\lambda U)$. The action in equation (A.7) is supersymmetric under the transformations

$$\begin{aligned}
\delta_\epsilon q^\mu &= \epsilon^*\psi^{*\mu} - \epsilon g^{\mu\nu}\psi_\nu\\
\delta_\epsilon \psi^{*\mu} &= -\mathrm{i}\epsilon[\dot q^\mu + \mathrm{i}g^{\alpha\beta}\Gamma^\mu_{\beta\gamma}\psi_\alpha\psi^{*\gamma} + \mathrm{i}g^{\mu\nu}(\nabla_\nu U)]\\
\delta_\epsilon \psi_\mu &= \mathrm{i}\epsilon^*[g_{\mu\sigma}\dot q^\sigma - \mathrm{i}\Gamma^\beta_{\mu\alpha}\psi^{*\alpha}\psi_\beta - \mathrm{i}(\nabla_\mu U)] - \epsilon g^{\alpha\beta}\Gamma^\gamma_{\mu\beta}\psi_\beta\psi_\gamma.
\end{aligned} \tag{A.8}$$

The generalisation to curved manifolds is straightforward. For $L = \frac{1}{2}g_{\mu\nu}\dot q^\mu\dot q^\nu - V(q)$ the corresponding ground-state wave equation is

$$H\psi_0 = -\tfrac{1}{2}\hbar^2 g^{\mu\nu}\nabla_\mu\nabla_\nu\psi_0 + V(q)\psi_0 = E_0\psi_0 \tag{A.9}$$

from which

$$V(q) = \tfrac{1}{2}g^{\mu\nu}(\nabla_\mu U)(\nabla_\nu U) - \tfrac{1}{2}\hbar g^{\mu\nu}(\nabla_\mu\nabla_\nu U) + E_0. \tag{A.10}$$

This leads to the supersymmetric path integral

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\ E_0\right)\int \mathrm{D}[q]\,\mathrm{D}[\psi^*]\,\mathrm{D}[\psi]\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\{\tfrac{1}{2}g_{\mu\nu}\dot q^\mu\dot q^\nu - \tfrac{1}{2}g^{\mu\nu}(\nabla_\mu U)(\nabla_\nu U)\right.$$
$$\left.\times\ \psi^{*\mu}\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\delta^\nu_\mu - \mathrm{i}\Gamma^\nu_{\mu\alpha}\dot q^\alpha - g^{\nu\tau}(\nabla_\mu\nabla_\tau U)\right]\psi_\nu + \tfrac{1}{2}R_\alpha{}^\beta{}_\gamma{}^\delta\psi^{*\alpha}\psi_\beta\psi^{*\gamma}\psi_\delta\}\right) \tag{A.11}$$

which is supersymmetric under the transformations of equation (A.8).

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